ISRAEL JOURNAL OF MATHEMATICS **132** (2002), 277–283

SYLOW NORMALIZERS AND CHARACTER TABLES, II

BY

GABRIEL NAVARRO*

Departament d'Àlgebra, Facultat de Matemàtiques
Universitat de València, 46100 Burjassot, València, Spain
e-mail: gabriel@uv.es

ABSTRACT

Suppose that G is a finite p-solvable group and let $P \in \mathrm{Syl}_p(G)$. In this note, we prove that the character table of G determines if $\mathbf{N}_G(P)/P$ is abelian.

1. Introduction

One of the classical problems in character theory is to determine what properties of a finite group G can be read off from its character table.

THEOREM A: *Suppose that G is p-solvable and let $P \in \mathrm{Syl}_p(G)$. Then the character table of G determines if $\mathbf{N}_G(P)/P$ is abelian.*

Our arguments heavily use a strong form of the Alperin–McKay conjecture for p-solvable groups which does not hold in general. Consequently, with our present approach we cannot decide if Theorem A is true or false for every finite group.

The results here are independent of those appearing in part I ([2]).

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* Research partially supported by DGICYT.
Received October 7, 2001

2. Preliminaries

As in [3], we fix M a maximal ideal of the ring of algebraic integers R containing pR, and we let *: $S \to F$ be the canonical ring homomorphism from $S = \{\alpha/\beta | \alpha, \beta \in R, \beta \in R - M\}$ onto the field $F = R/M$. Also, S is a local ring with maximal ideal $\mathcal{P}$. (See Chapter 2 of [3] for more details.) Our notation for blocks also follows [3].

We start with an elementary lemma.

(2.1) Lemma: *Suppose that G has a normal Sylow p-subgroup P and let $K = \mathbf{O}_{p'}(G)$. Let Θ be a complete set of representatives of orbits of the G-action on* $\mathrm{Irr}(K)$.

(a) *The sets* $\mathrm{Irr}(G|\theta)$, *where $\theta \in \Theta$, are all the different blocks of G.*

(b) *Suppose that G/P is abelian. If $\theta \in \Theta$, then the number of p'-degree irreducible characters of G lying over θ does not depend on θ. The same happens with the number of irreducible Brauer characters of G lying over θ.*

Proof: By the Schur–Zassenhaus theorem, we have that $\mathbf{C}_G(P) = \mathbf{Z}(P) \times K$. Let B be a block of G. Now, by Corollary (9.3) of [3], we have that all $\mathrm{Irr}(B)$ lie over a unique $\theta \in \Theta$. If T is the stabilizer of θ in G, by the Fong–Reynolds Theorem (9.14) of [3], we have that there is a unique block b of T such that $\mathrm{Irr}(b) \subseteq \mathrm{Irr}(T|\theta)$ such that $\mathrm{Irr}(B) = \{\psi^G | \ \psi \in \mathrm{Irr}(b)\}$. Also $P \subseteq T$. Therefore $\mathbf{O}_{p'}(T) \subseteq \mathbf{C}_G(P)$ and we conclude that $\mathbf{O}_{p'}(T) = K$. By Fong's theorem (10.20) of [3], we have that $\mathrm{Irr}(b) = \mathrm{Irr}(T|\theta)$, and therefore $B = \mathrm{Irr}(G|\theta)$ by the Clifford correspondence (Theorem (6.11) of [1]). Finally, given $\eta \in \Theta$, there is a block covering η (for instance, by Theorem (9.2) of [3]), and part (a) easily follows.

For part (b), notice that if G/P is abelian, then θ extends to G. (Since $K \cap P = 1$, we may see θ as a character of G/P.) Let $\nu \in \mathrm{Irr}(G)$ be any extension of θ. By Gallagher's theorem (Corollary (6.17) of [1]), we have that $\mathrm{Irr}(G|\theta) = \{\nu\delta | \delta \in \mathrm{Irr}(G/K)\}$. Therefore, the number of p'-degree irreducible characters of G lying over θ is the number of p'-degree irreducible characters of G/K.

Finally, since the irreducible Brauer characters of G have P in their kernel, we see that the number of those lying over θ is just the number of irreducible characters of G lying over $1_P \times \theta$. This number is $|G : PK|$. ∎

In the next result, we obtain information on the character table of groups G in which $\mathbf{N}_G(P)/P$ is abelian. As we will point out, this is not enough to guarantee a converse.

(2.2) THEOREM: *Let G be a finite group, let $P \in \mathrm{Syl}_p(G)$ and assume that $\mathbf{N}_G(P)/P$ is abelian.*

(a) *Assume the Alperin–McKay conjecture for G. If B is a p-block of G of maximal defect and B_0 is the principal block, then the number of irreducible characters of p'-degree of B and B_0 coincide.*

(b) *Whenever $x \in G$ is a p-regular element of G having conjugacy class size not divisible by p and $\chi \in \mathrm{Irr}(G)$ has p'-degree, then $\chi(x) \neq 0$ mod $\mathcal{P}$.*

Proof: Write $\mathrm{Irr}_{p'}(B)$ for the p'-degree characters of the block B of G. Let $b \in \mathrm{Bl}(\mathbf{N}_G(P))$ be the Brauer first main correspondent of B. Also, let b_0 be the principal block of $\mathbf{N}_G(P)$. Then, by the Alperin–McKay conjecture, we have that $|\mathrm{Irr}_{p'}(B)| = |\mathrm{Irr}_{p'}(b)|$ and $|\mathrm{Irr}_{p'}(B_0)| = |\mathrm{Irr}_{p'}(b_0)|$. By Lemma (2.1.b), we have that $|\mathrm{Irr}_{p'}(b)| = |\mathrm{Irr}_{p'}(b_0)|$, and the first part of the theorem follows.

Suppose now that x is p-regular such that $P \subseteq \mathbf{C}_G(x)$. Let $K = \mathrm{cl}_G(x)$. By Lemma (4.16) of [3], we have that $K \cap \mathbf{C}_G(P)$ is a conjugacy class of $\mathbf{N}_G(P)$. Now, since $\mathbf{N}_G(P)/P$ is abelian, notice that $x \in \mathbf{Z}(\mathbf{N}_G(P))$. Therefore, we have that $K \cap \mathbf{C}_G(P) = \{x\}$. Let $\chi \in \mathrm{Irr}(G)$ be of p'-degree, and assume that $\chi \in \mathrm{Irr}(B)$, where B is a block of G of maximal defect. Also, let $b \in \mathrm{Bl}(\mathbf{N}_G(P))$ be the Brauer correspondent of B. Since $b^G = B$, we have that

$$\lambda_B(\hat{K}) = \lambda_b(x).$$

Now, let $\psi \in \mathrm{Irr}(b)$. Since $x \in \mathbf{Z}(\mathbf{N}_G(P))$, we have that $\psi(x) = \psi(1)\epsilon$, where ϵ is a p'-root of unity. Now,

$$(|K|\chi(x)/\chi(1))^* = (\psi(x)/\psi(1))^* = \epsilon^* \neq 0.$$

Now, since $|K|_p = \chi(1)_p = 1$, we conclude that $\chi(x)^* \neq 0$ mod $\mathcal{P}$, as desired. ∎

The converse of Theorem (2.2) is not in general true (even if we assume conditions (a) and (b) for every factor group of G). For instance, if G is a simple group having a unique block of maximal defect (the principal block), then the only p-regular class of p'-size of G is the identity. (This is a well-known fact; see, for instance, Problem (15.8) of [1].) In this case, the hypotheses of Theorem (2.2) are automatically satisfied, and $\mathbf{N}_G(P)/P$ need not be abelian. (For instance, $G = Th$ for $p = 7$.)

If G is p-solvable, we have more information on Brauer correspondent blocks (and eventually, this is what makes our Theorem A true).

(2.3) THEOREM: *Suppose that G is p-solvable. Let $P \in \mathrm{Syl}_p(G)$ and assume that $\mathbf{N}_G(P)/P$ is abelian.*

(a) *If B is a p-block of G of maximal defect and B_0 is the principal block, then the number of irreducible Brauer characters of p'-degree of B and B_0 coincide.*

(b) *Whenever $x \in G$ is a p-regular element of G having conjugacy class size not divisible by p and $\varphi \in \mathrm{IBr}(G)$ has p'-degree, then $\varphi(x) \not\equiv 0 \bmod \mathcal{P}$.*

Proof: Write $\mathrm{IBr}_{p'}(B)$ for the p'-degree irreducible Brauer characters of the block B. Let $b \in \mathrm{Bl}(\mathbf{N}_G(P))$ be the Brauer first main correspondent of B. Also, let b_0 be the principal block of $\mathbf{N}_G(P)$. Then, by Theorem (2.2.ii) of [4], we have that $|\mathrm{IBr}_{p'}(B)| = |\mathrm{IBr}_{p'}(b)|$ and $|\mathrm{IBr}_{p'}(B_0)| = |\mathrm{IBr}_{p'}(b_0)|$. By Lemma (2.1), we have that $|\mathrm{IBr}_{p'}(b)| = |\mathrm{IBr}_{p'}(b_0)|$, and part (a) follows.

Now, suppose that $\varphi \in \mathrm{IBr}(G)$ has p'-degree. By the Fong–Swan theorem, let $\chi \in \mathrm{Irr}(G)$ be a lifting of φ. Then $\varphi(x) = \chi(x)$ and part (b) follows from Theorem (2.2.b). ∎

3. Proof of Theorem A

This is the precise algorithm that enable us to decide from the character table of G if the group $\mathbf{N}_G(P)/P$ is abelian.

(3.1) THEOREM: *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Then $\mathbf{N}_G(P)/P$ is abelian iff for every factor group $\bar{G}$ of G the following conditions are satisfied:*

(a) *If B is a p-block of $\bar{G}$ of maximal defect and B_0 is the principal block of $\bar{G}$, then the number of irreducible Brauer characters of p'-degree of B and B_0 coincide.*

(b) *Whenever $x \in \bar{G}$ is a p-regular element of $\bar{G}$ having conjugacy class size not divisible by p and $\varphi \in \mathrm{IBr}(\bar{G})$ has p'-degree, then $\varphi(x) \not\equiv 0 \bmod \mathcal{P}$.*

In order to prove Theorem (3.1), we need one more lemma.

(3.2) LEMMA: *Suppose that $N \triangleleft G$ with G/N abelian.*

(a) *Let $\theta \in \mathrm{Irr}(N)$. Then $|\mathrm{Irr}(G|\theta)| = |G : N|$ iff θ extends to G.*

(b) *Suppose that N is abelian. Then every $\theta \in \mathrm{Irr}(N)$ extends to G iff G is abelian.*

Proof: (a) If θ extends to G, this is clear by Gallagher's theorem (Corollary (6.17) of [1]). For the converse, let T be the stabilizer of θ in G. Then

$$|G : N| = |\mathrm{Irr}(G|\theta)| = |\mathrm{Irr}(T|\theta)|$$

by the Clifford correspondence. By Problem (11.10) of [1], we have that $|\operatorname{Irr}(T|\theta)|$ is the number of certain special conjugacy classes of T/N. In particular,

$$|\operatorname{Irr}(T|\theta)| \leq |T:N| \leq |G:N|,$$

and we deduce that θ is G-invariant. Now, since

$$|G:N| = \sum_{\chi \in \operatorname{Irr}(G|\theta)} (\chi(1)/\theta(1))^2,$$

and $|\operatorname{Irr}(G|\theta)| = |G:N|$, we deduce that $\chi(1) = \theta(1)$ for all $\chi \in \operatorname{Irr}(G|\theta)$. This proves part (a).

(b) If G is abelian, then every $\theta \in \operatorname{Irr}(N)$ extends to G. Assume now that every $\theta \in \operatorname{Irr}(N)$ extends to G. Then every $\theta \in \operatorname{Irr}(N)$ is G-invariant, and it follows from Clifford's theorem that

$$\operatorname{Irr}(G) = \bigcup_{\theta \in \operatorname{Irr}(N)} \operatorname{Irr}(G|\theta)$$

is a disjoint union. Hence, the number of conjugacy classes of G is

$$k(G) = |N||G:N| = |G|,$$

and we deduce that G is abelian. ∎

Proof of Theorem (3.1): Suppose first that $\mathbf{N}_G(P)/P$ is abelian. If $N \triangleleft G$, then

$$\mathbf{N}_{G/N}(PN/N)/PN/N \cong \mathbf{N}_G(P)N/PN \cong \mathbf{N}_G(P)/P\mathbf{N}_N(P)$$

is abelian. Now, Theorem (2.3) applied to the factor groups of G proves one direction of Theorem (3.1).

Suppose now that for every factor group $\bar{G}$ of G we have that the conditions (a) and (b) are satisfied. We prove that $\mathbf{N}_G(P)/P$ is abelian by induction on $|G|$. Hence if $1 < N \triangleleft G$, we have that $\mathbf{N}_G(P)/P\mathbf{N}_N(P)$ is abelian by induction. In particular, we may assume that $\mathbf{O}_p(G) = 1$.

Now, let $N = \mathbf{O}_{p'}(G) > 1$ and recall that $\mathbf{N}_G(P)/P\mathbf{N}_N(P)$ is abelian. We claim that $\mathbf{N}_N(P) = \mathbf{O}_{p'}(\mathbf{N}_G(P))$. Of course, $\mathbf{N}_N(P) \subseteq \mathbf{O}_{p'}(\mathbf{N}_G(P))$. However, $\mathbf{O}_{p'}(\mathbf{N}_G(P)) \subseteq N$, by a standard result on p-solvable groups, and the claim follows. Now, let Θ be a complete set of representatives of $\mathbf{N}_G(P)$-orbits on $\operatorname{Irr}(K)$, where $K = \mathbf{N}_N(P)$. Given $\theta \in \Theta$, let $B_\theta = (b_\theta)^G$, where b_θ is the unique block of $\mathbf{N}_G(P)$ whose irreducible characters lie over θ (by Lemma (2.1)). By the Brauer First Main theorem, we have that the B_θ's are all the blocks of G

of maximal defect. Now, by hypothesis, and Theorem (2.2.ii) of [4], we have that $|\,\mathrm{IBr}_{p'}(b_\theta)| = |\,\mathrm{IBr}_{p'}(b_1)|$. Now, since the irreducible Brauer characters of $\mathbf{N}_G(P)$ contain P in their kernel, we have that $|\,\mathrm{IBr}_{p'}(b_\theta)| = |\mathrm{Irr}(\mathbf{N}_G(P)|1_P \times \theta)|$. Therefore, we conclude that

$$|\mathrm{Irr}(\mathbf{N}_G(P)|1_P \times \theta)| = |\mathrm{Irr}(\mathbf{N}_G(P)|1_{P\times K})|.$$

Since $\mathbf{N}_G(P)/PK$ is abelian, we have that

$$|\mathrm{Irr}(\mathbf{N}_G(P)|1_P \times \theta)| = |\mathbf{N}_G(P) : P \times K|.$$

By Lemma (3.2.a), we conclude that the character $1_P \times \theta$ extends to $\mathbf{N}_G(P)$ for every $\theta \in \mathrm{Irr}(K)$.

Now, we claim that K is abelian. Otherwise, there exists $\theta \in \mathrm{Irr}(K)$ with $\theta(1) > 1$. By Burnside's theorem on zeros, we may find $x \in K$ such that $\theta(x) = 0$. Notice that $[P, x] = 1$ and x is p-regular. Let C be the conjugacy class of x, so that $|C|_p = 1$. Also, $L = C \cap \mathbf{C}_G(P)$ is a conjugacy class of $\mathbf{N}_G(P)$ (by Lemma (4.16) of [3]) with $|L|_p = 1$. Now, let $\psi \in \mathrm{Irr}(\mathbf{N}_G(P))$ be an extension of θ. In particular, ψ is a height zero character of the block b_θ. Also, $\psi(x) = 0$. Now, let φ be an irreducible Brauer character of height zero in B_θ, and let $\chi \in \mathrm{Irr}(B_\theta)$ be a lifting of φ. Now, by the Brauer correspondence, we have that

$$|C|\chi(x)/\chi(1) = |L|\psi(x)\psi(1) = 0 \bmod \mathcal{P}.$$

Since $|C|_p = \chi(1)_p = 1$, we deduce that

$$\chi(x) = 0 \bmod \mathcal{P}.$$

Therefore,

$$\varphi(x) = 0 \bmod \mathcal{P},$$

and this is a contradiction which proves the claim.

Hence, we have that PK/P is an abelian group, with $\mathbf{N}_G(P)/KP$ abelian, such that all irreducible characters of PK/P extend to $\mathbf{N}_G(P)$. By Lemma (3.2.b), we conclude that $\mathbf{N}_G(P)/P$ is abelian, as desired. ∎

Next is Theorem A of the Introduction.

(3.3) THEOREM: *Suppose that G is p-solvable and let $P \in \mathrm{Syl}_p(G)$. Then the character table of G determines if $\mathbf{N}_G(P)/P$ is abelian.*

Proof: We have to prove that the character table of G ($\mathrm{ct}(G)$) determines the conditions (a) and (b) of Theorem (3.1). First of all, $\mathrm{ct}(G)$ determines the character table of its factor groups. So it suffices to show that $\mathrm{ct}(G)$ determines (a) and (b) for G. First of all, $\mathrm{ct}(G)$ determines the different p-blocks of G. (Perhaps, the easiest way to do that is to apply Theorem (3.19) of [3]. Recall that $\mathrm{ct}(G)$ determines if an element is p-regular by Higman's theorem (8.21) of [1].) Now, the blocks of maximal defect are those having a p'-degree irreducible character. Now, by Corollary (10.4) of [3], we can detect the irreducible Brauer characters of G of p'-degree. Finally, we can easily check condition (b) from the character table. ∎

Notice that by the proof of Theorem (3.1), we only need to check conditions (a) and (b) for those factor groups of the form G/N where N is a member of the $\mathbf{O}_{pp'}$-series.

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